How much number theory do you have to know to be a sunflower?
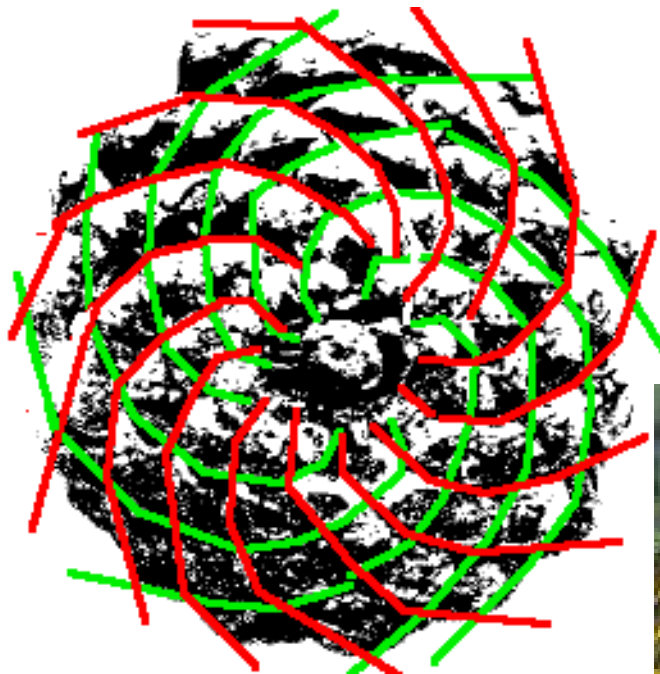
Leonid Levitov (MIT)

Kiev 23.10.2015

Fibonacci numbers in plant morphology

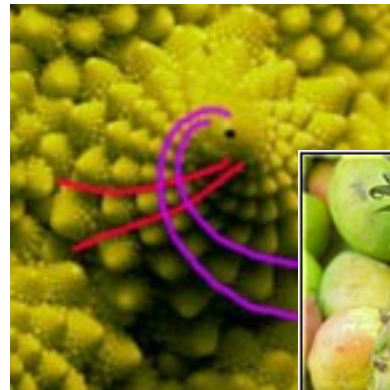1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144...

Fibonacci phyllotaxis:
numbers of spirals (parastichies)
are consecutive Fibonacci pairs
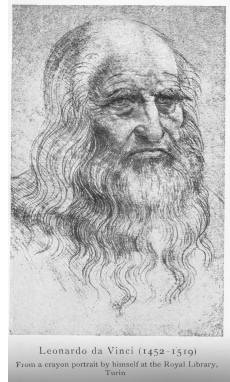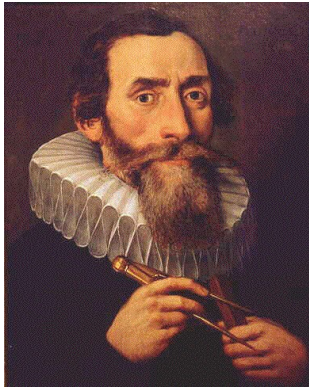

Leaf arrangement (Goethe)

(8, 13)

Long history: observed, characterized, systematized

Leonardo Da Vinci (Notebook 1503):

    Nature has arranged the leaves of the latest branches of many plants so that the sixth is always above the first, and so it follows in succession if the rule is not impeded.

and many, many many others

R. V. Jean (J. Theor. Biology, 1978):

    The fascinating question: "*Why does the Fibonacci sequence arise in the spirals seen in plants?*" seems to be at the heart of problems of plant morphology. In atomic physics, Balmer's series opened the way to Bohr's theory of the atom and then to quantum mechanics. The great hope of bio-mathematicians is that one day they may be able to do for biology what has been done by mathematical physicists in physics.

Non-Fibonacci numbers?

Yes, but also special:

Lucas numbers

$$1, 3, 4, 7, 11, 18, 29, 47, 76...$$

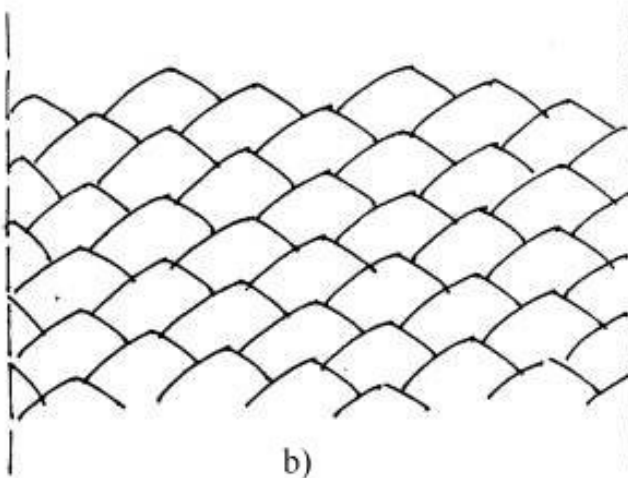Statistics for cones of pine-trees (Norway):
  95% Fibonacci
  4% Lucas
  1% deficient

Models of phyllotaxis
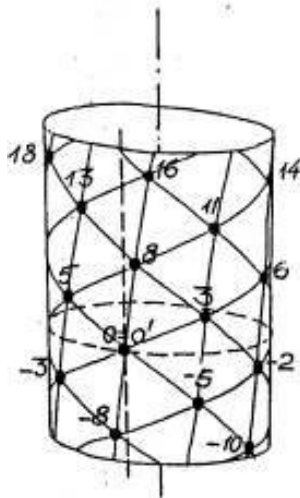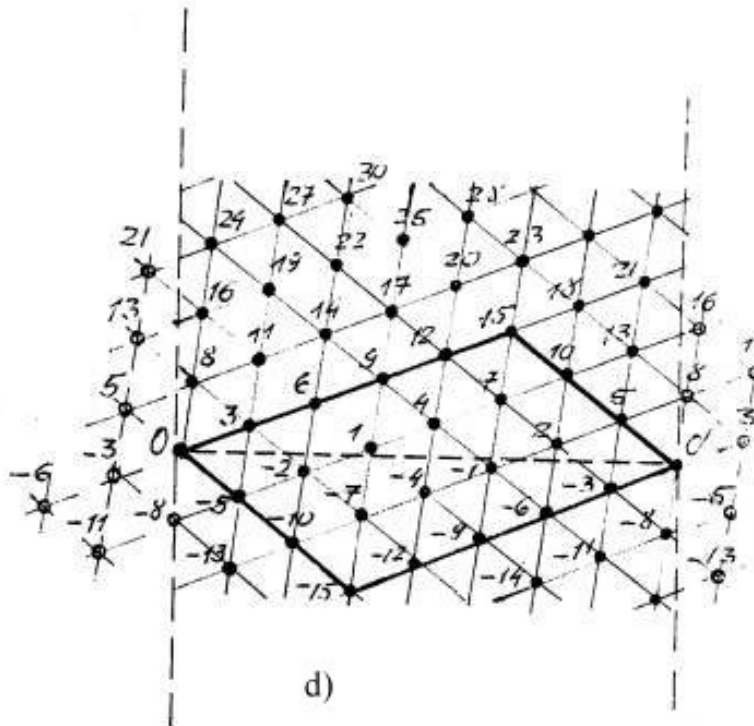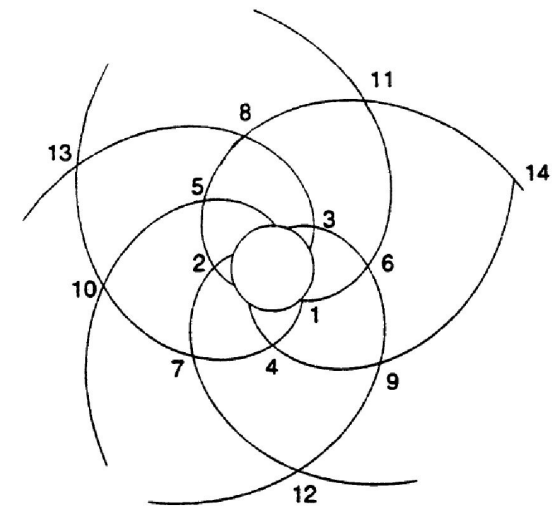

Cylindrical lattices
(cones, pineapples,
seed heads, etc)

Spiral lattices
(not today)

Geometry of cylindrical lattices
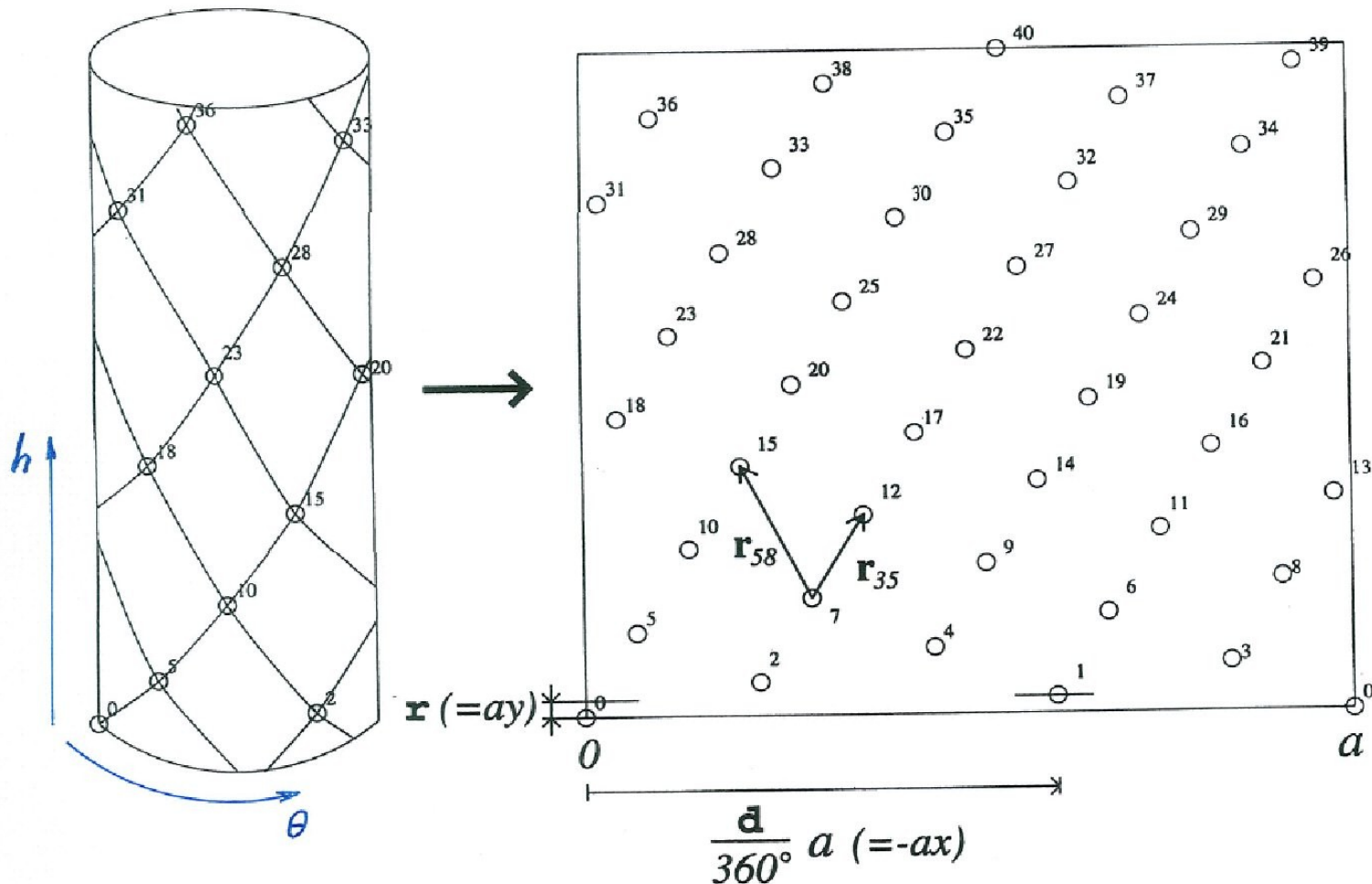
Generating helix $h_m = \mathrm{r}\,m, \quad \theta_m = \mathrm{d}\,m$

Parastichies: lattice rows
defined by **shortest vectors**

Parastichy type of a lattice: (N, M)
right left

Lattice phase space (x,y)

A more convenient parameterization: $\mathbf{r} = u\mathbf{i} + v\mathbf{j}$ (Cartesian system)

$$\mathbf{r}_{pm} = u_{pm}\mathbf{i} + v_{pm}\mathbf{j} = a\left((p - mx)\mathbf{i} + my\mathbf{j}\right) = \sqrt{A}\left(\frac{p - mx}{\sqrt{y}}\mathbf{i} + m\sqrt{y}\mathbf{j}\right)$$

Unit cell area (A=1)

Given x and y, what are the parastichy numbers N and M?

Parastichy domains in the x, y plane: domains of constant N, M

Boundaries are arcs of circles (*lattices with rectangular unit cell*) N=n, M=m;

All mutually prime N, M theoretically possible!
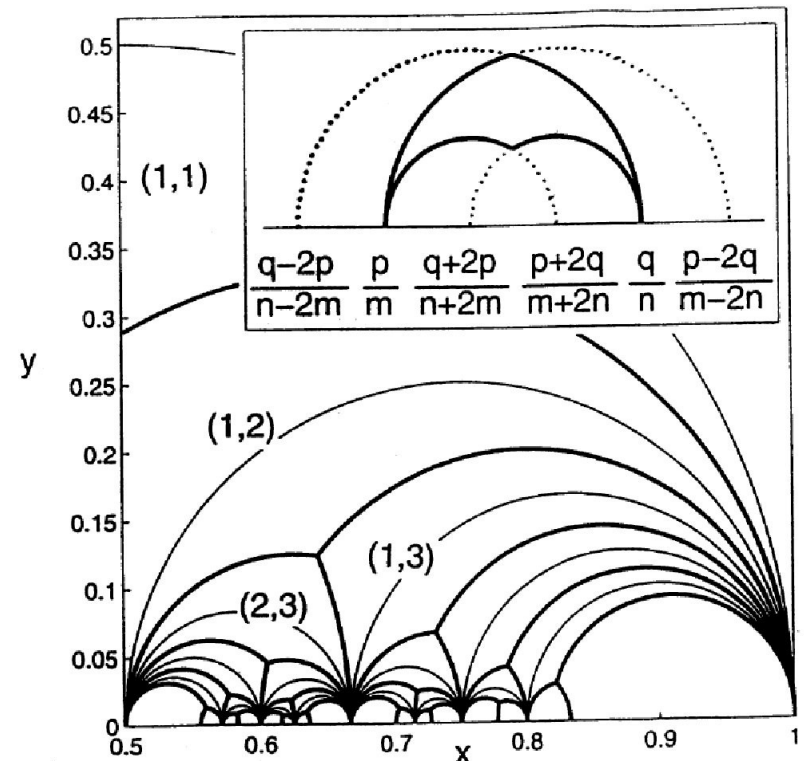
An interesting example: close-packed disks on a cylinder

Lattices with rhombic unit cell: a Cayley tree in phase space

Densely packed (triangular) lattices

(m,n), m<n

(n,m+n)   (m,m+n)

(1,1)

(1,2)   (1,2)

(1,3)   (2,3)   (2,3)   (1,3)

(3,5)   (2,5) (2,5)   (3,5)

r

0°   90°   180°   270°   360°

d
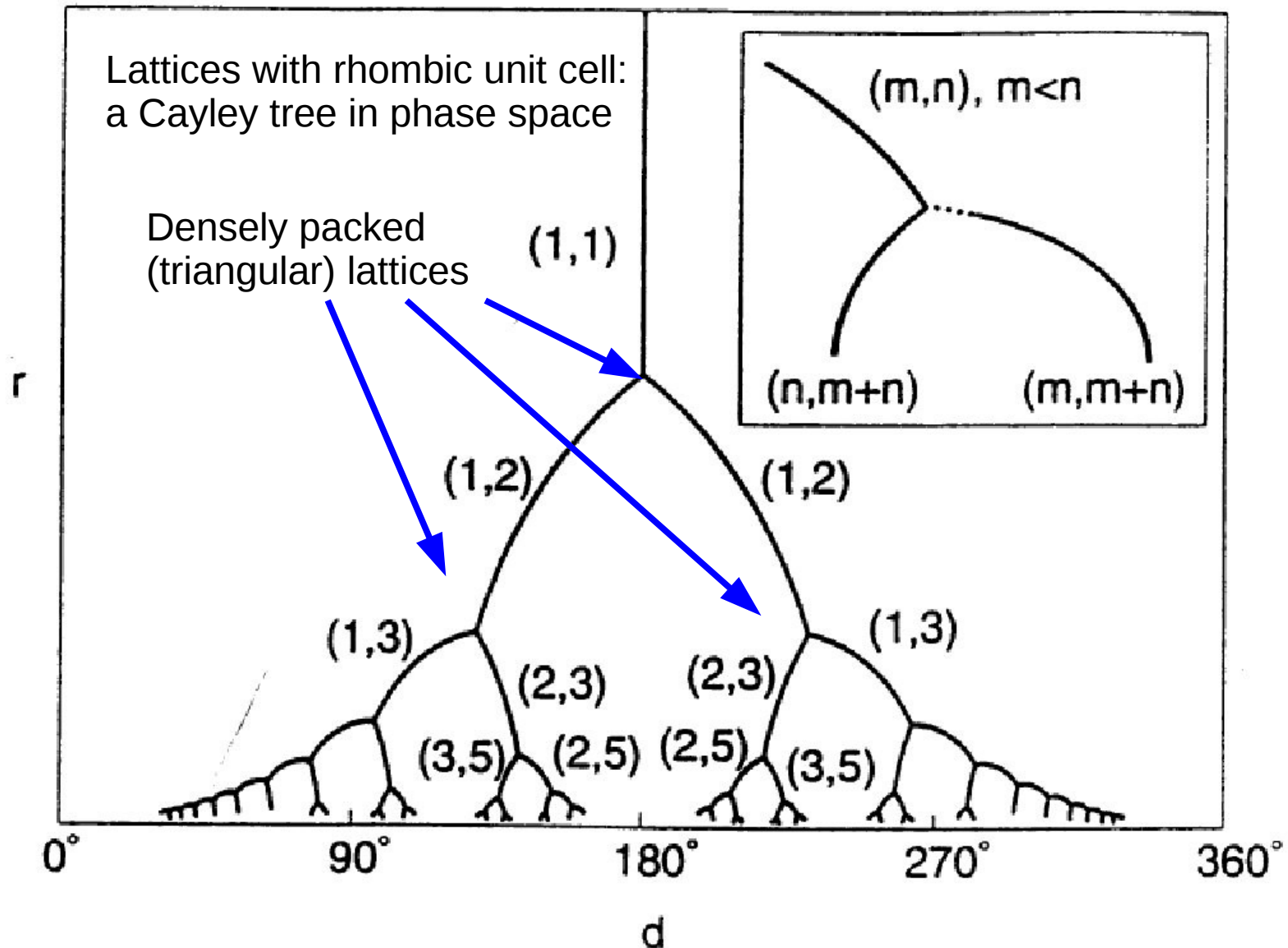
What have we learned so far?

Cylindrical lattices are a useful model (phase space, parastichy numbers, etc)

Hints at connection with hyperbolic geometry: Cayley tree

Does not explain the predominant occurrence of Fibonacci numbers, since all N,M possible

Mechanical theory of phyllotaxis

Energy model: growth under stress, phyllotactic patterns result from the development of deformation

$$E_{\text{total}} = \frac{1}{2} \sum_{pmp'm'} U(|\mathbf{r}_{pm} - \mathbf{r}_{p'm'}|).$$

The repulsive interaction U(r) models contact pressure between neighboring structural units (scales, seeds, etc) during growth

For example: $U(r) = U_0 e^{-r/r_0}$, or $U(r) = U_0/|r|^{\gamma}$, or $U(r) = U_0 e^{-r^2/r_0^2}$

Claim: anisotropic growth (slow axial, fast radial) deterministically generates Fibanacci phyllotactic patterns

Trajectories in the phase space

History of growth-induced deformation

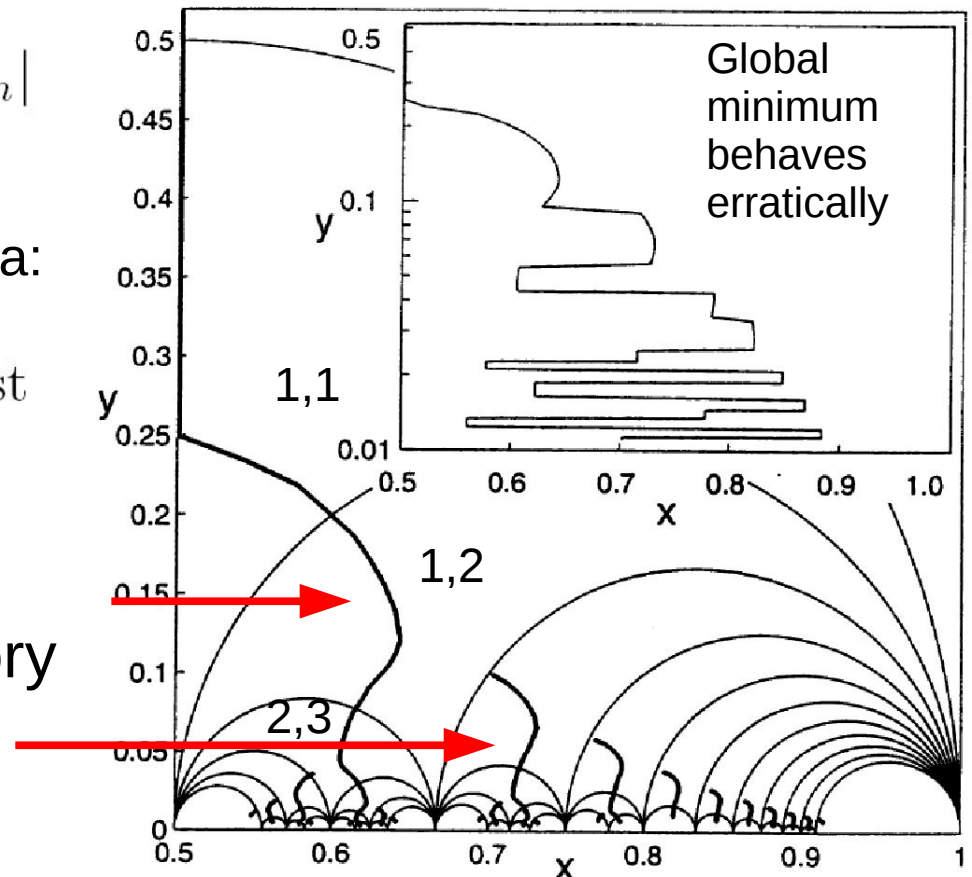$$E(x,y) = \sum_{pm} U(r_{pm}), \quad r_{pm} = |\mathbf{r}_{pm}|$$
$$A = 1$$

Track positions of local energy minima:

$$\frac{\partial E}{\partial x} = 0, \quad \frac{\partial^2 E}{\partial x^2} > 0, \quad A, \; y = \text{const}$$

Striking observation: principal trajectory tracks all Fibonacci domains, next principle trajectory yields Lucas numbers!

Fibonacci phyllotaxis obtained from a deterministic process!

We see that

By varying y and tracking energy minima all Fibonacci patterns are obtained one by one in a deterministic manner;

Other trajectories give generalized Fibonacci sequences (e.g. Lucas);

This behavior is robust, results do not depend on the choice of potential $U(r)$, provided it is repulsive.

Experimental realization: magnetic cactus

of hard spheres [9]. To properly test Levitov's model, we constructed two versions of a *magnetic cactus* consisting of 50 outward-pointing, dipolar permanent magnets (spines) mounted on stacked coaxial bearings free to rotate about a vertical axis (stem), as in Fig. 1. The system can be annealed into a lower-energy state by mechanical agitation
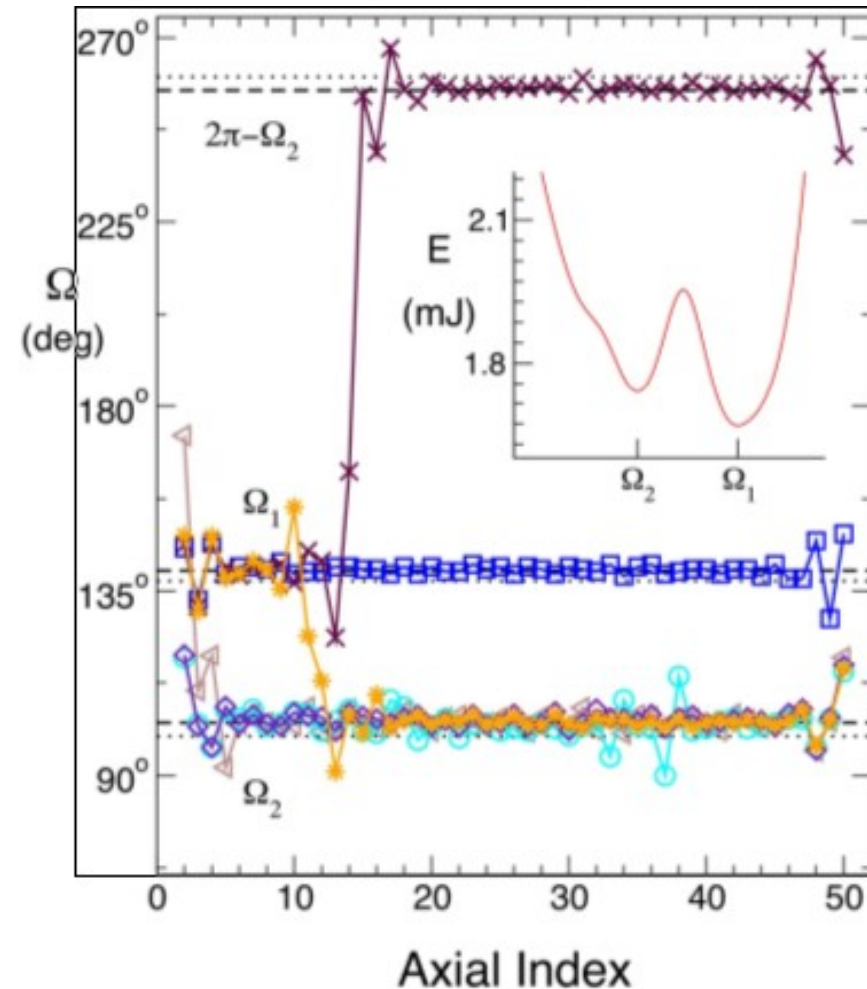
Static and Dynamical Phyllotaxis in a Magnetic Cactus

Cristiano Nisoli,[1,3] Nathaniel M. Gabor,[2,3] Paul E. Lammert,[3] J.D. Maynard,[3] and Vincent H. Crespi[3]

[1]CNLS and T-Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA
[2]Department of Physics, Cornell University, 109 Clark Hall, Ithaca, New York 14853-2501, USA

Think hyperbolic (model has analytic solution)

Interpret the x,y plane as a hyperbolic plane

Define curvilinear triangles with vertices
p/m, q/n, (p+q)/(m+n)   (|pm-nq|=1)

THEOREM: The trajectories of the energy minima behave the same way in all triangles for repulsive *U(r)*

Proof relies on GL(2,Z) symmetry of E(x,y):
Use z=x+iy to define modular transformation

$$z'=(az+b)/(cz+d)$$

with integer a,b,c,d such that |ad-bc|=1, then E(x',y')=E(x,y), where z'=x'+iy'.

Farey triangles partition the x,y plane into fundamental domains of GL(2,Z)
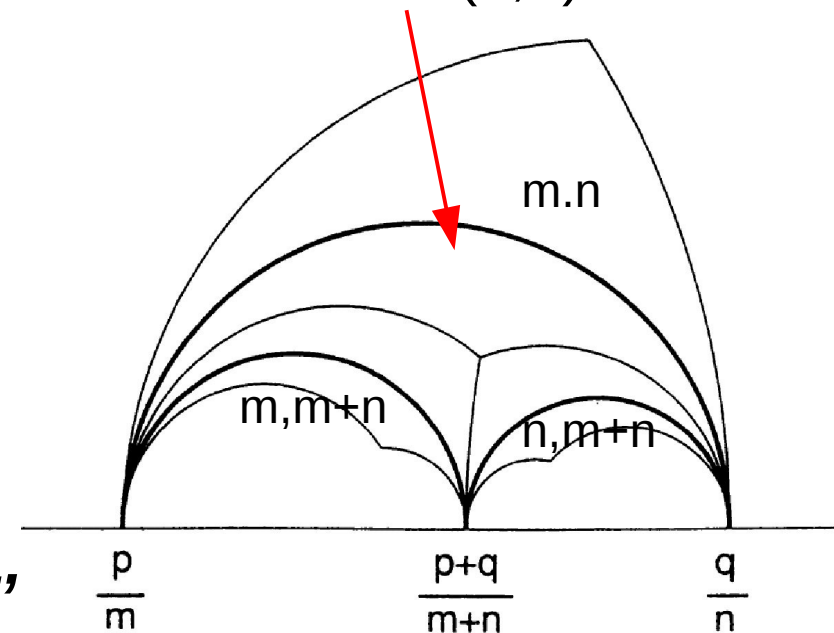
Energy E(z) landscape in the hyperbolic plane

E(z) is invariant of the modular group

Minima of E(z): perfect triangular lattices

Saddle points of E(z): square lattices

Each triangle is a valley of E(z) surrounded by three ridges with three passes
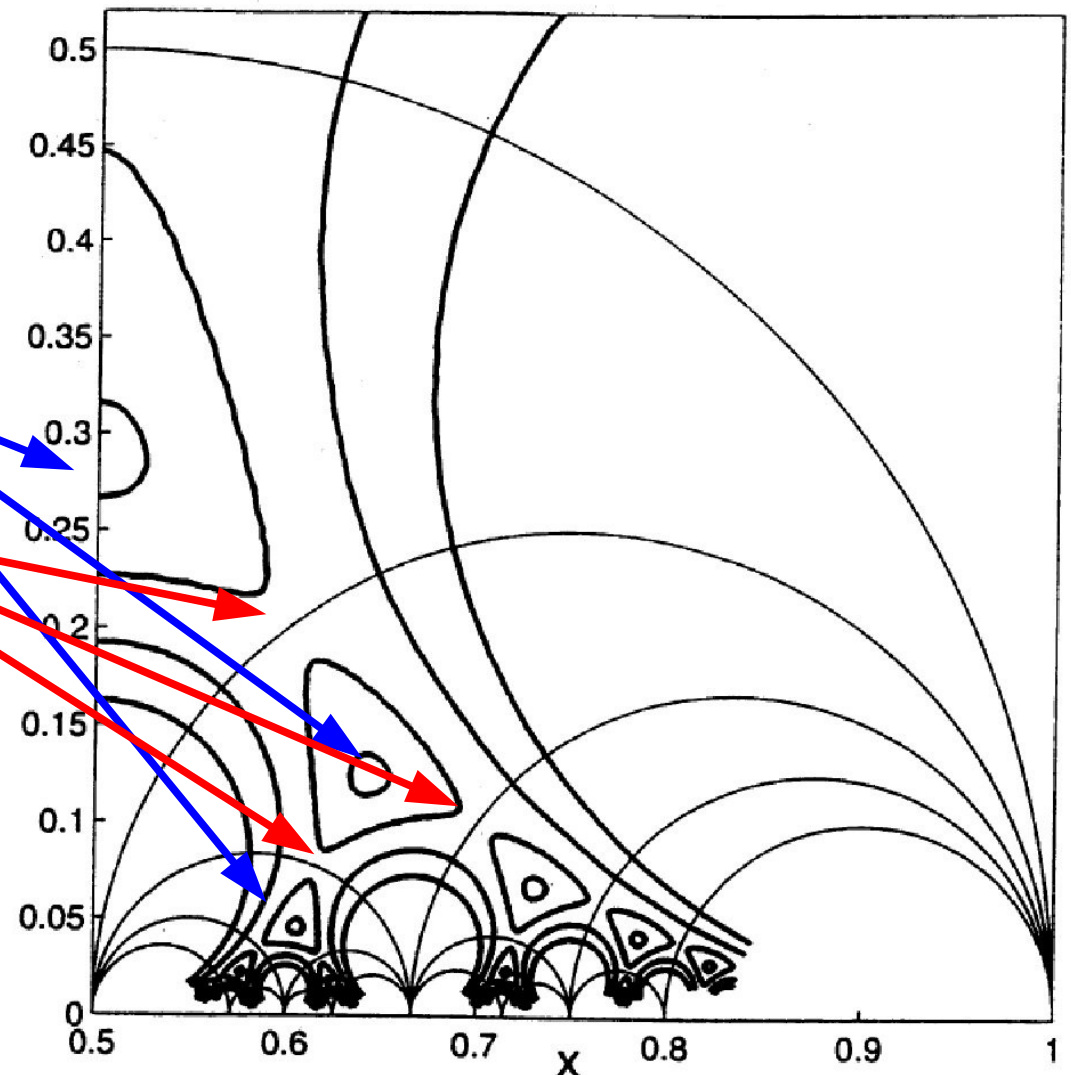
Trajectories in one fundamental domain

Since E(z) is GL(2,Z)-invariant,
it is sufficient to analyze behavior
in just one fundamental domain
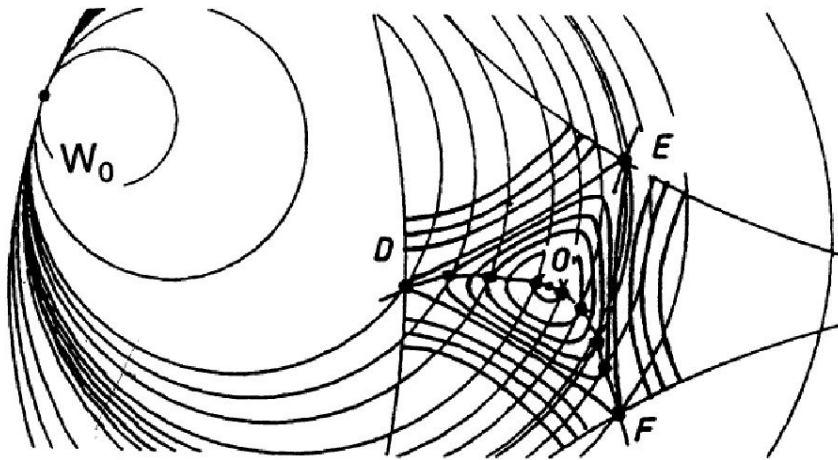
(b)
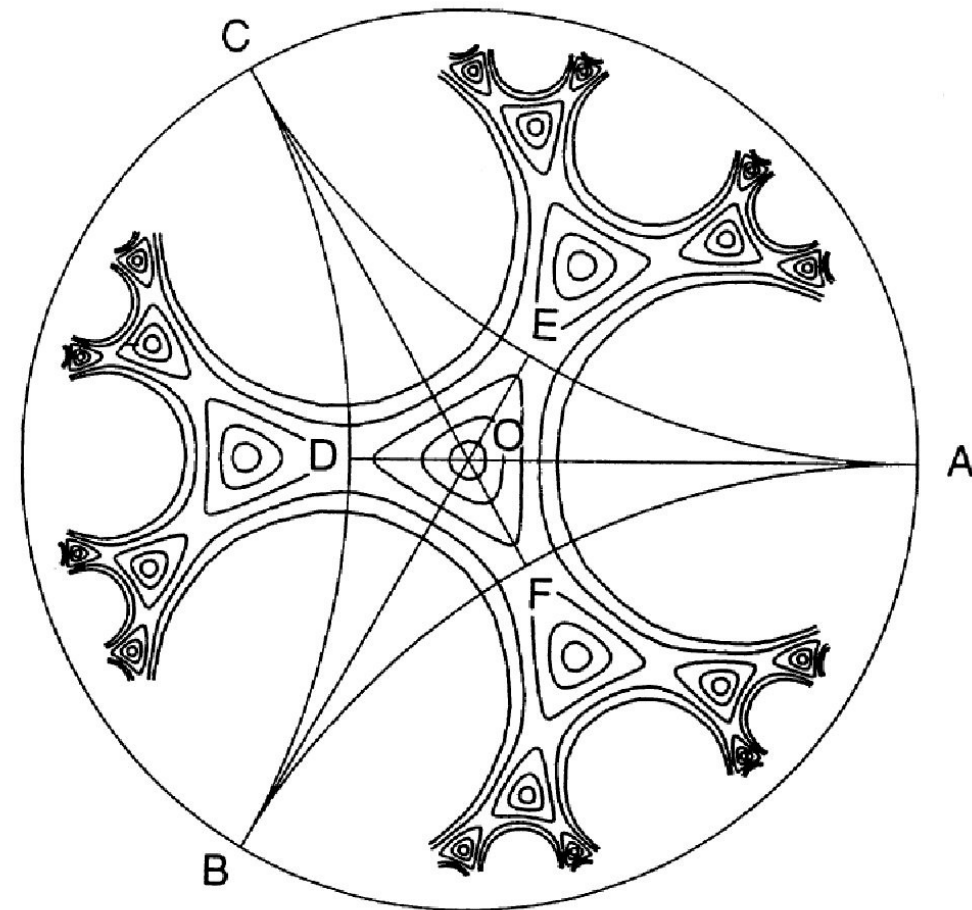
(a)

Connect trajectories in all triangles:
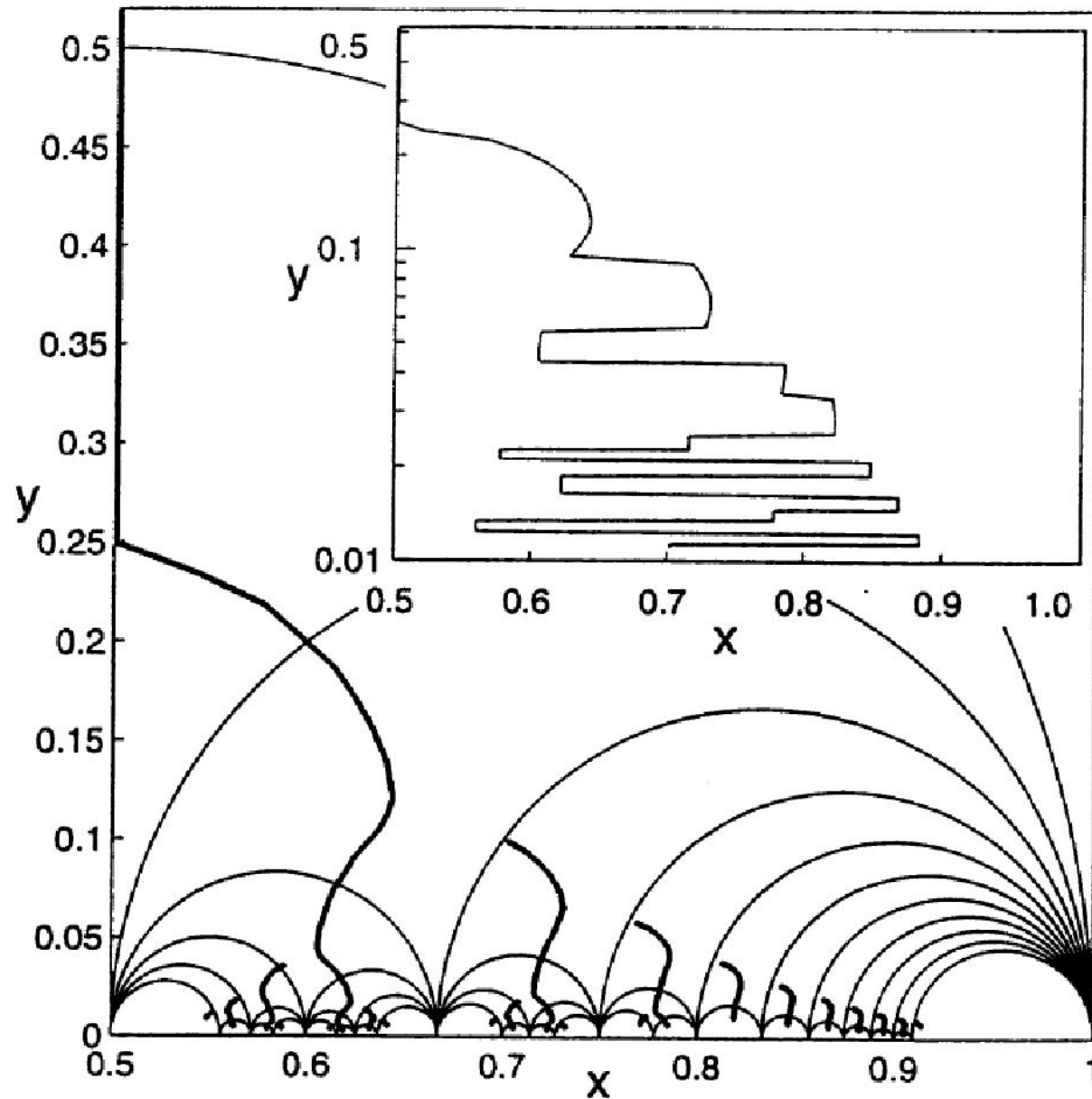
Summing up

Determinism: No bifurcations (except one where left/right symmetry is lost). Finite gaps between different trajectories

Principal trajectory: Fibonacci sequence

All other trajectories described by generalized Fibonacci sequences

One mistake (loss of continuity) gives Lucas sequence, the most common exception;

This behavior is robust, claim victory: Fibonacci phyllotaxis explained

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Enrichment programs for highschool students, bachelor and master students

Leonid Levitov (levitov@mit.edu)

Education & Students & Research as seen from MIT-Physics

- **Research University model: many benefits over traditional models**
- **Research labs on campus, open to ALL students**
- **Research experience is a vital part of education**
- **Toy research morphs into real research**
- **Education and research are integrated (via students)**

Education & Students & Research:
UROP program

- From a faculty perspective, UROP students are one of the best things about MIT. Undergraduates who spend time in your lab, working on research. Full of creativity and energy. Can be the spark that lights a fire. And they don't cost much
- From an undergraduate student perspective, UROP-ing is one of the best things to do at MIT. Real research experience. Jump in and then learn to swim. Sense of accomplishment. And you get paid, too
- Guiding UROP students toward careers in nnovation and research, in industry or at universities
- Network w/ universities that have UROP e.g. Imperial UK & ETH Zurich; summer exchange of UROP students
- Exchange master students (much like UROP)

MIT Summer programs

MIT does not offer a traditional open-enrollment summer school program where any high school student can come to campus to take courses and live in the dorms. However, several partner organizations run small, specialized programs on campus. If you'd rather study the human genome or build a robot than memorize this year's summer TV reruns, then you might try one of these:

Research Science Institute (RSI) brings together about 70 high school students each summer for six stimulating weeks at MIT. This rigorous academic program stresses advanced theory and research in mathematics, the sciences and engineering. Participants attend college-level classes taught by distinguished faculty members and complete hands-on research, which they often then use to enter science competitions. Open to high school juniors, the program is free of charge for those selected.

Women's Technology Program (WTP) is a four-week summer academic and residential experience where 60 female high school students explore engineering through hands-on classes (taught by female MIT graduate students), labs, and team-based projects in the summer after their junior year. Students attend WTP in either Electrical Engineering and Computer Science (EECS) or Mechanical Engineering (ME).

MIT Launch - a 4-week entrepreneurship program for high school students, teaching the entrepreneurial skills and mindset through starting real companies. Students go through rigorous coursework, collaborate with peers and mentors, and use the multitude of tools surrounding them at MIT to realize what it takes to be successful in the real world – resourcefulness, adaptability, and innovation. Many need-based scholarships are available.

While the Summer Science Program (SSP) is not on campus, MIT-co-sponsored science research program. With locations in New Mexico and Colorado, and many MIT students among the program's alumni/ae, students learn mathematics, physics, astronomy, and programming over the program's 6 weeks. The curriculum is organized around a central research project: to determine the orbit of a near-earth asteroid (minor planet) from direct astronomical observations

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Note:
1) All programs are (highly) competitive but buy your lottery ticket
2) Some are officially `for high school students' but in fact also accept international undergrads;
3) Some require $$ but selection is on merit, never money-based
4) Some are free to all selected students and some even pay you

and innovation. Many need-based scholarships are available.

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Education & Students & Research:
International summer schools

- Boulder Summer School (Master, PhD level)
- MPI Dresden (PhD&Master)
- Les Houches (PhD&Master)
- Weizmann Summer program (international undergrads)
- Windsor Summer School (Master&PhD)
- Also: ICTP Trieste, Dynastia (?)
- Deadlines, $$

Other selective Summer programs

Most summer programs admit all or most students who can pay the (high) tuition. However, a number of competitive-admission summer programs select only the best students on the basis of merit and are often free or comparatively affordable. MIT offers four of our own (above); here are a few more:

Science & Research Programs

BU Research Internship Program

Clark Scholar Program

Garcia Summer Scholars

High School Summer Science Research Program (HSSSRP)

High School Honors Science/Mathematics/Engineering Program (HSHSP)

International Summer School for Young Physicists (ISSYP)

Secondary Student Training Program (SSTP)

Stanford Institutes of Medicine Summer Research Program (SIMR)

Student Science Training Program (SSTP)

QuestBridge College Prep Scholarship

Math Summer programs

AwesomeMath

Canada/USA Mathcamp

Hampshire College Summer Studies in Mathematics (HCSSiM)

Honors Summer Math Camp (HSMC)

MathILy

Program in Mathematics for Young Scientists (PROMYS)

The Ross Program

Stanford University Mathematics Camp (SUMaC)

Prove It! Math Academy